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THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE
INTERESTS OF TEACHERS OF MATHEMATICS

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CONTENTS

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INTERESTS OF TEACHERS OF MATHEMATICS

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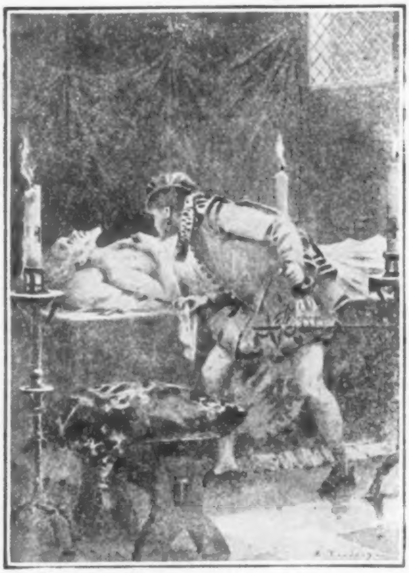

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THE MATHEMATICS TEACHER

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VOLUME III DECEMBER, 1910 NUMBER 2

SPECIAL DEVICES IN TEACHING GEOMETRY.

By Paul Noble Peck.

Since the colleges have adopted a more or less uniform standard of entrance requirements, necessitating a high degree of efficiency on the part of the candidates for admission, the burden of preparation falls upon the secondary schools. This raising of the standard on the part of the colleges frequently sorely taxes these schools by requiring a greater amount of work in preparatory subjects without any proportionate increase in the time supposed to be devoted to such preparation. Many of our high schools and private preparatory schools today are doing much of the work that, a decade ago, was considered the function of the first-year college classes. To complete the additional work imposed by this increase in the entrance requirements without slighting any of the subjects is by no means a simple task. The work must be, to a certain extent, condensed and simplified, or special devices resorted to as a means of arousing the student's interest or directing and conserving his energies.

In the multitudinous array of subjects in which entrance examinations are now offered, mathematics, since it is required of all students, occupies a most prominent place. It must be said of many of the preparatory schools, that they are meeting in a satisfactory manner the demand made upon them by the colleges and technical schools for well-trained students in this subject.

The object of this paper is not to criticise the general training in mathematics given in the secondary schools, but rather to offer, in the special topic of geometry, suggestions which the writer trusts may be of assistance in lightening the constantly increasing burden of preparation.

From my personal experience in the class-room I shall briefly summarize the most important deficiencies of the average college freshman class in geometry.

There is a marked tendency on the part of the student to keep close to the text, to draw the figure and to letter it as it is in the book. This carries with it the effort to memorize, in part, at least, the exact words of the proof, particularly in all propositions relating to or in any way involving the theory of limits. I have frequently requested a student who has given a letter-perfect book proof to erase his figure and to draw it again in a different position and with other lettering, and in nearly all cases with the same result—the pronounced discomfiture of the student due to his inability to proceed! Such cases of course are the result of memorizing the text and figure without any clear idea of the relation of the particular proposition to the others that have preceded it. The student who attempts by this means to go through the ordeal of the day's recitation is invariably the one who sooner or later informs you, in a sudden burst of confidence, that he does not like mathematics but his particular forte is French or German!

Another serious fault is inaccuracy in definitions and carelessness in constructions. When a student is told that he should not memorize his proof, but rather make it his own before giving it, he usually goes to the other extreme in his definitions and constructions, with occasionally surprising results. The three following will illustrate my point. One student, upon being asked to define a right angle, stated that it was the angle formed by two perpendicular lines, and when further asked when two lines are perpendicular, replied, "When they are at right angles." Another student, a middle-aged man, was requested to let fall a perpendicular to a line from an outside point. His reply, given in all seriousness, was, "Let go of it"! A third student was asked to draw a tangent to a circle from a point without the circle, and the amazing reply was, "It can't be done." When

asked why the problem was, to his way of thinking, impossible, he surprised the class by explaining that " You can't draw a tangent to a circle if the circle isn't there "! Evidently, from the very nature of their replies, these students had no clear understanding of the peculiar function of geometry and were not impressed with the need of accuracy in statement or construction.

The fault here lies in the very beginning of the students' training in the subject. Had they acquired the proper geometrical view-point, such answers, of course, would not have been given. So then, the responsibility for the student's proper progress in geometry rests with the preparatory schools. If a student complete his course with but a hazy conception of what is to be expected of him when he continues his subject in college, he is doomed at best to trail along at the end of his class and may consider himself fortunate if he is rated among the " simply passed."

The next point I shall consider is the tendency of students, when given a choice between numerical problems and general theorems, to select the former. This has been such a general experience in my freshman classes that I am forced to conclude that the average student prefers the solution by formula to any that would in the slightest degree tend to test his ability to reason for himself. It is so much easier to substitute numerical values for the letters in the formula and then turn the crank and await results! Not that I undervalue the numerical problem, for I believe that it is an essential in the application of theory to practice, but this preference on the part of the student is an evidence of lack of confidence in his reasoning powers and should be early overcome.

The average freshman, I have discovered, has little or no knowledge of the historical side of mathematics and is completely bewildered if asked to explain, for example, the difference between the Pythagorean Proposition and the Pons Asinorum. There may be advanced the argument that the history of mathematics is peculiarly a topic for study in the college course. Certainly the preparatory school is burdened and has scarcely enough time as it is, without being called upon to teach this additional topic in the already long list of subjects

required for college entrance. It seems to me, however, that much of the important historical data can be given the student at the time he meets with propositions of historical interest, and in this way he is enabled to absorb the valuable facts without much conscious effort.

I shall now briefly discuss a few of the faults in the method of teaching geometry in vogue in certain schools. There are many teachers who are in the habit of reading out the theorems to students as they are sent to the board, one after another, and by the time the third student is reached, the first one is back again with the request that his theorem be restated to him. This is apt to be followed by similar requests from other students and the result is a general confusion lasting sometimes for five or ten minutes. The loss of time thus occasioned is chiefly responsible for another and a graver fault. I refer to the acceptance, by teachers who are anxious to finish the assigned lesson in the period usually allotted to it, of proofs by "previous propositions." Is there a teacher who has not had pupils eager to conceal their ignorance of the particular lemmas required in the demonstration of their propositions by falling back upon that much-worn and overburdened phrase, "by a previous proposition"? If so, I have yet to meet him. To the lazy pupil this is a "thank-goodness," an oasis in a long and dreary desert! And yet there are teachers who not only permit their students to recite in this slip-shod manner, but I know of one instance where students were actually required to remember that a particular statement was true by section 245, for instance, or by paragraph 317, and this without even referring back to the section or paragraph quoted to familiarize themselves with the wording therein! What is the natural consequence of such a method? Aside from the reflection upon the teacher, it engenders carel ess of the worst sort in the pupil and if persisted in can result only in failure.

Another fault frequently encountered is the failure of the student to connect the subject matter of today's lesson with that of yesterday. This lack of continuity is a serious matter particularly in geometry and should have special attention.

I have mentioned a few of the deficiencies of students as they come to me from the high school and preparatory schools and

also the more important faults in the method of teaching used in some institutions. What steps shall be taken to remedy these conditions? It is my purpose to describe to you briefly a few devices that I am using in my own classes with excellent results. With these time is saved, confusion eliminated, accuracy of statement and a clearer understanding of the continuity of geometrical proof obtained, and in every case the student's interest is aroused and he is made to think for himself.

In the first place I have introduced the card catalogue into geometry. The text of each proposition is put on a card together with its section number as given in the text-book, and a complete index of the lemmas employed in each proposition is made by section number and filed on cards. This is for the teacher. A record-sheet is prepared covering the assignment for the day. At the top of this sheet is the record of attendance. Then follows a list of the assigned propositions and under each of these is the list of lemmas employed in its proof. All of this list is by section number and each number is followed by two blank spaces, one to be filled in with the name of the student and the other with his mark which should be recorded as soon as the recitation is finished.

In assigning the propositions to the class, I call upon the first student, hand him the card with his theorem on it and send him to the board. The card contains nothing but the text of the theorem and the section number. While he is going to the board, the next pupil is called and sent to the board with his card. This is continued for the entire assignment. While the students to whom assignments have been made are at the board drawing the figures and arranging the details of their proofs, the rest of the class can be quizzed on definitions or from charts, which I shall mention shortly. In this way every available minute is utilized and every member of the class participates in some way in the recitation each day. This in itself is quite an important detail. When the students at the board have finished drawing their figures, they one by one return the cards to the desk and take their seats. They are then called upon for the recitations, beginning with the one to whom has been assigned the most fundamental lemma. When he has finished the student who is responsible for the theorem employing this lemma is called upon

and after him the student having the proposition next higher in order and so on until the main proposition is reached. When the proof of this theorem is given, there is no need for the student to fall back upon the uncertain "previous proposition" phrase for the previous propositions are all before the class and have been discussed. This gives the class a constant review of all the text in the way it is most needed, by showing in just what way the propositions are related and upon what they depend and why this dependence exists.

I have referred to certain charts used in quizzing the class. The preparation of these I believe to be the most valuable work the student can do and by this means he acquires a most intimate knowledge of all the relations of the geometrical propositions. The student is required to prepare an index similar to the one mentioned above, showing by section number every lemma used in the proof of each proposition. Corollaries and definitions are not included, only the general propositions being used. After this has been properly verified, a geometrical tree is prepared for each book. The texts of the fundamental propositions, that is, those not requiring lemmas, are written at the bottom of a standard size piece of cardboard and enclosed in a rectangular frame. The section number corresponding to the one in the book is attached, together with the proper figure lettered as in the book or otherwise at the teacher's pleasure. Above these are placed the propositions requiring the fundamental theorems in their proof, and over these are still others requiring the second set and so on as the tree grows. Finally, at the top of the chart, in the case of the first book, are the last propositions of that book. In the charts of the other books, at the top, will be found the propositions of the particular book for which the chart is prepared and under these the dependent propositions. These framed texts are properly connected by straight lines showing the dependence of each proposition on every proposition subordinate to itself. Thus the student has before him on a single chart not only every relation between the propositions of a particular book, but the relations of all the propositions that have preceded these, as well. This gives him the entire chain and the continuity is so impressed upon him that he cannot fail to understand at a glance the inter-relation between the theorems.

On the historical side, my plan has been to assign to individual members of a class topics of interest to be carefully prepared and read to the class. With each paper is required a complete bibliography. In this way the student's interest is aroused and even the student who does not care for mathematics begins to find some pleasure in his work and to hope for the best when the examinations come!

It has been my experience in the use of all these devices that the difficulties mentioned in the first part of this paper have, in a large measure, been overcome and the work has been presented to the student in a pleasing manner and one calculated to stimulate an earnest effort on his part.

They are therefore offered in the hope that they may assist other teachers as they have the writer.

THE GEORGE WASHINGTON UNIVERSITY,
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As thoughts possess the fashion of the mood
  That gave them birth, so every deed we do
Partakes of our inborn disquietude
  Which spurns the old and reaches towards the new.
The noblest works of human art and pride
Show that their makers were not satisfied.

For looking down the ladder of our deeds,
  The rounds seem slender; all past work appears
Unto the doer faulty; the heart bleeds,
  And pale Regret comes weltering in tears,
To think how poor our best has been, how vain,
Beside the excellence we would attain.

—HENRY ABBEY.

THE ASSOCIATION OF TEACHERS OF MATHEMATICS IN THE MIDDLE STATES AND MARYLAND.

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2. To generalize correctly; and,

3. To develop originality and insight by using skillfully a finely worked out language of symbols.

It is believed that the body of closely related truths in algebra are admirably adapted to the purpose just mentioned. This is not to say that examples and illustrations to provide interesting exercise in algebraic practice are not to be drawn from commerce, geometry, physics, mechanics; but such examples should be so simple as to require no extended explanation of their nature and whatever knowledge they imply should be regarded as incidental to the main purpose of cutivating the student's powers of reasoning. The first concern of the young student of algebra is the knowledge, logic, and operations contained within algebra itself, but it is perfectly possible to select attractive exercises which, while they do not carry the student far afield, will show him how algebra can be practically applied.

As the same idea occurs in different forms in various parts of the algebra, some repetition is unavoidable in the syllabus, but perhaps it is most desirable to recur several times to a method that does not change though it be applied to a form that is new.

Throughout the syllabus are notes bearing upon particular points, and at the close are appended a few general notes which are suggestive of the many observations the teacher has to make and of the caution his work constantly requires.

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I. Extension of Arithmetic in Algebra.

A. Literal numbers as the generalization of arithmetic numbers.

1. Indicated operations, $a+b$; $a-b$; $a\times b$; $a\div b$.
2. Powers and fractions resulting from indicated multiplications and divisions.
3. Negative numbers necessary for a more complete scale of numbers. In arithmetic $3-10$ is an impossible

operation. It becomes possible as soon as negative numbers are admitted. Introduce the scale . . . -3, -2, -1, 0, $+1$, $+2$, $+3$, . . . by addition and subtraction. Illustrate by divisions on a line, and by as many concrete examples as possible.

4. Simple problems involving the use of literal numbers, *e. g.*, John has a cents and received b cents, he spends x cents; how much has he? A letter stands for a number which may be integral, fractional, positive, or negative.

B. Simple equations.

1. Contrast 30% of cost = \$60 with $.3x = 60$.
   $5 + 7 = 12$. $19 - 3 = 22 - 6$; substitute a letter in these examples.
2. Solution of $3x - 4 = x + 8$ by the use of the equality axioms.
3. Discover law of transposition.
4. Literal equations.
   a. $x + a = b$
   b. $dx + b = c$
   c. More easy problems.
5. Some very simple problems resulting in numerical simultaneous equations.

Note 1. Extract definitions as they are needed.

Note 2. Graphs of simple forms such as

$$y = 2x + 1, \text{ and } \begin{cases} y = 2x + 1 \\ y = 3x + 2 \end{cases}$$

Use coördinate paper. Measure the x and y of intersection.

II. Fundamental Operations.

Note.—Introduce the laws of signs and of exponents and the laws of commutation, association, and distribution as applied to these operations. Some work in detached coefficients should be given.

A. Addition and Subtraction.

1. Algebraic addition involves arithmetic addition and subtraction.
2. Meaning of subtraction is to find a number which, added

to the subtrahend, gives the minuend. Check subtraction by addition.

3. Removal and introduction of signs of aggregation. Simple cases only.

B. Multiplication.

1. Monomials by monomials.
2. Polynomials by monomials.
3. Polynomials by polynomials.

Note 1.—Notion of *function* and *variable* may be suggested in this place by the evaluation of a polynomial for different values of the letter in it.

Note 2.—Laws of homogeneity may be pointed out.

4. Type products.

a. $(x \pm y)^2$, and the square of any polynomial.

b. $(x+y)(x-y)$, $(x+y+z)(x+y-z)$,
$(x^2+xy+y^2)(x^2-xy+y^2)$

c. $(cx+a)(cx+b)$

d. $(a \pm x)^3$

e. Of the form, $(a \pm b)(a^4 \mp a^3b + a^2b^2 \ldots)$.

Note.—It should be pointed out that the given expressions are the factors of the results.

C. Division.

1. Monomial divisors.
2. Polynomial divisors.
3. Special quotients of the type: $\dfrac{(x^n \pm y^n)}{x \pm y}$.

Note 1.—Binomial divisors and simple synthetic division are especially recommended at this point.

Note 2.—The Remainder Theorem may be introduced here by simple numerical examples. It may again be referred to in connection with factoring.

The simple proof by the identity $f(x) \equiv Q(x-a) + R$ may well be given here.

III. Factoring.

Note.—Refer to II. B, 4 and II. C, 3.

1. Common Factor.
   a. $ax + bx$.
   b. $a(x + y) + b(x + y)$
   c. Grouping, as $ax + ay + bx + by$.
2. Perfect square, $x^2 \pm 2ax + a^2$.
3. Difference of squares, $x^2 - a^2$, $x^2 \pm 2ax + a^2 - b^2$, $x^4 + x^2y^2 + y^4 = (x^4 + 2x^2y^2 + y^4) - x^2y^2$
4. Quadratic trinomial, $x^2 + ax + b$, $ax^2 + bx + c$.
5. Perfect cube, $x^3 \pm 3x^2y + 3xy^2 \pm y^3$.
6. Sum or difference of odd powers, $x^3 \pm y^3$.
7. Binomial factors by the remainder theorem or synthetic division. Simple cases only.

   Note 1.—Above are the type forms to be led up to by many simple numerical examples.

   Note 2.—Equations that can be solved by factoring, as $x^2 - 3x = 10$, $ax + bx + c = 0$, should be introduced here.

IV. Highest Common Factor and Lowest Common Multiple. [By Factoring.]

1. Monomials only:
   $60a^2x^2$, $45a^4x^3$, $20ax^4$.
2. Polynomials only:
   a. $x^3 - 8$, $x^2 - 4$, $x^2 + 5x - 14$.
   b. $x^2 - ax - bx + ab$, $x^2 - 2bx + b^2$, $x^2 - b^2$.
3. Products of monomial and polynomial.
   $10a^2x^2 - 90a^2$, $15a^3x^2 + 30a^2x - 225a^3$.

V. Fractions.

Note.—A whole number is a fraction with the denominator 1.

A. Reduction to lowest terms.

1. Monomial numerator and denominator:

$$\frac{49a^5x^3}{35a^7x^2} = \frac{7\cdot 7\cdot a^5\cdot x^2\cdot x}{7\cdot 5\cdot a^5\cdot x^2\cdot x^2} = \frac{7\cdot a^5\cdot a^2}{7\cdot a^5\cdot x^2}\cdot\frac{7x}{5a^2} = \frac{7x}{5a^2}.$$

Then show the cancellation method.

2. Polynomial numerator and denominator:

   a. Type, $\dfrac{(x-7)(x+7)}{(x-7)(x-5)}$, $\dfrac{(x-a)(x+b)}{5(x-a)}$, etc.

b. Type, $\dfrac{(x-5)(x+3)}{a(5-x)}$, $\dfrac{(x-a)(x-b)}{(b-x)(b+x)}$

Note.—What is cancellation? Emphasize $a - a = 0$,

$$\frac{a}{a} = 1, \quad \frac{a-b}{a-b} = ? \quad \frac{a-b}{b-a} = ?$$

B. Reduction to mixed expressions and the reverse. Many examples, especially of the latter.

C. Addition and subtraction of fractions.

1. Reducing fractions to a common denominator.
   a. Monomial.
   b. Binomial and polynomial.
2. Addition and subtraction.
   a. Monomial denominators.
   b. Polynomial denominators.

(1) Type, $\dfrac{m+n}{m-n} - \dfrac{m-n}{m+n} - \dfrac{4m^2}{m^2-n^3}$

(2) Type, $\dfrac{b+c}{(a-b)(a-c)} + \dfrac{c+a}{(b-c)(b-a)} + \dfrac{a+b}{(c-a)(c-b)}$

Note.—Examples, in which the sum can be reduced to lower terms, are especially recommended.

D. Multiplication and division of fractions.

1. Monomial numerators and denominators.
2. Polynomial numerators and denominators.

Note 1.—Arithmetical processes with fractions such as multiplying and dividing both terms by the same number, reducing to the same common denominator, etc., should be recalled and compared with the corresponding operations in algebra. For each of these processes a sufficient reason should be given and appeal should be made to the student's intuition.

Note 2.—The first elementary notion of ratio may be introduced with fractions and the forms $\dfrac{a}{\infty}, \dfrac{0}{a}, \dfrac{a}{0}, \dfrac{0}{0}$, explained with such simple numerical examples as, $\dfrac{2}{\infty}, \dfrac{0}{2}, \dfrac{2}{0}, \dfrac{x^2-1}{x-1}$, when $x=1$.

E. Simplification of complex fractions.

1. The continued fraction (simple types only).

$$\cfrac{1}{1 + \cfrac{1}{x + \cfrac{1}{x}}}$$

2. Complex fractions having a sum or difference in both numerator and denominator.

$$\frac{\dfrac{1}{a-b} - \dfrac{a}{a^2-b^2}}{\dfrac{a}{ab+b^2} - \dfrac{b}{a^2+ab}}$$

3. Mixed numbers in both numerator and denominator.

$$\frac{\dfrac{2mn}{m+n} - 1}{1 - \dfrac{n}{m+n}}, \quad \frac{2x - 1 - \dfrac{10}{x-3}}{3x - 5 + \dfrac{12}{x-2}}$$

Note.—Multiplying both terms of the fraction by the same expression is often the simplest device. Examples should not be too complex and some should be workable in several ways.

VI. The Simple Equation.

A. Extended practice in the simple equation, resuming the consideration of transposition.

B. The fractional equation:

1. Monomial denominators:

$$\frac{x-1}{7} = 7 - \frac{4+x}{4} - \frac{23-x}{5}$$

2. Polynomial denominators:

a. $$\frac{2x+1}{3x-3} = \frac{7x+1}{6x-6} - \frac{2x^2-3x-45}{4x^2-4}$$

b. $$\frac{1}{x-1} - \frac{1}{x-2} = \frac{1}{x-3} - \frac{1}{x-4}$$

Note.—Simplify the members in *b* separately before clearing of fractions.

c. $$\frac{6x+1}{15} - \frac{2x-4}{7x-16} = \frac{2x-1}{5}$$

Note.—Care must be taken that extraneous roots are not introduced in clearing of fractions.

C. The literal equation:

1. Solution for one unknown.
2. "Formula work." Solving simple formulæ for each of the letters in terms of the others.

Note.—Only formulæ that can be *briefly* explained should be chosen, as:

a. $i = \frac{ptr}{100}$, (simple interest)

b. $d^2 = 2s^2$, (diagonal of a square)

c. $h = \frac{s}{2}\sqrt{3}$, (height of equilateral triangle)

d. $C = (F - 32)\frac{5}{9}$, (Centigrade scale)

e. $s = vt$

$c = 2\pi r$

$i = \frac{l}{d^2}$, (light), etc., giving numerical illustrations.

D. Problems leading to simple equations.

1. Oral work, as: the sum of a and $b = ?$ When a is divided by b, c is the quotient, and d the remainder; express algebraically. A train goes at the rate of m miles an hour for h hours, how far does it go? The interest on a dollars at b per cent. for c years is what?
2. Problems dealing with—
   a. Geometrical objects.
   b. Motion (trains, etc.)
   c. Number.
   d. Business.
3. Some simple abstract problems, illustrated by concrete examples.

VII. Equations in Two and Three Unknowns.

A. Two unknowns.

Elimination by, 1. Addition and subtraction.
2. Comparison.
3. Substitution.

Note.—Some equations should be solved by all three ways for the sake of comparing the methods.

B. Two unknowns, fractional form:

$$\frac{4}{5x}+\frac{5}{6y}=5\frac{11}{15}$$

$$\frac{5}{4x}-\frac{4}{5y}=\frac{11}{20}$$

C. Three unknowns.

1. Where all three equations contain all three unknowns; with small numerical coefficients only.
2. Where the three unknowns are not in each equation.

$$\frac{4}{x}-\frac{3}{y}=\frac{1}{20}$$

$$\frac{2}{z}-\frac{3}{x}=\frac{1}{15}$$

$$\frac{4}{z}-\frac{5}{y}=\frac{1}{12}$$

D. Literal equations.

$$(a-b)x-(a+b)y=-4ab$$
$$(a+b)x+(a-b)y=2a^2-2b^2$$

E. Problems leading to equations in two and three unknowns.

F. Plotting easy equations in two unknowns, such examples as give:

1. Lines intersecting at any angle.
2. At right angles.
3. Lines that are parallel, pointing out the relations between the coefficients.

VIII. Involution and Evolution.

A. Powers of a monomial. The general form $(a^m)^n$ carefully examined and illustrated.

B. Square, cube and fourth power of a binomial.

C. Definition of root. Principal root. Roots of a monomial.

D. Square root of:

1. Polynomial, presented through the trinomial.
2. Numbers.

Note.—Derive the general method by the type form $a^2+2ab+b^2$.

IX. Exponents and Radicals.

A. Exponents in fundamental operations. (Refer to II).

B. Theory of a positive integral exponent.

Theorems:

1. $x^a \cdot x^b = x^{a+b}$
2. $x^a \div x^b = x^{a-b}$ $\quad (a > b)$
3. $(x^a)^b = x^{ab}$
4. $(x^a)^b = (x^b)^a$
5. $(x^a y^b)^n = x^{an} y^{bn}$
6. $\left(\frac{x^a}{y^b}\right)^n = \frac{x^{an}}{y^{bn}}$
7. $(\sqrt[n]{x^a})^n = x^a$

Note.—In considering the above theorems it is sufficient to use numerical exponents, the exponents being the abbreviations of continued multiplication. Restrict a and b to positive integers, $a > b$.

C. Negative, zero, and fractional exponents:

Note.—Apply the principle of *no exception* to interpret these new exponential forms.

Theorems:

1. $\frac{a^m}{a^n} = \frac{a^{m-n}}{1} = \frac{1}{a^{n-m}} \quad \frac{1}{a^{-(m-n)}}$

The scale, . . . $a^{-4}, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3$ . . .

Work in fundamental operations with these exponents.

2. When $n = m$, $\frac{a^m}{a^n} = \frac{a^m}{a^m} = a^{m-m} = a^0 = 1$

3. *a.* $x^{\frac{a}{n}} = \sqrt[n]{x^a}$ $\quad$ *cf.* B. 7

b. $\sqrt[n]{x^a} \cdot \sqrt[n]{y^b} \cdot \sqrt[n]{x^a \cdot y^b}$

Note.—All these theorems should be first indicated by numerical examples.

D. *Note.*—Radicals should be closely associated with exponents wherever possible.

1. The fractional exponent and radical sign interchanged and compared in application.
2. The graphic representation of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ by successive right-angled triangles.

E. Operations:

1. Removal of a perfect power: $\sqrt{18a^3b^5} = 3ab^2\sqrt{2ab}$
2. Reduction to lower degree: $\sqrt[6]{9a^2b^4} = \sqrt[3]{3ab^2}$, or vice versa.
3. Introduction of a factor under the radical sign: $2a\sqrt[3]{3b} = \sqrt[3]{24a^3b}$
4. The four fundamental operations and powers.

$$2\sqrt{a} + 3\sqrt{a} = 5\sqrt{a},\ \sqrt{a}\cdot\sqrt{b}\cdot\sqrt{c} = \sqrt{abc},\ \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}};$$

$(\sqrt{abc})^2$ etc. It is well to state some of these laws in words.

5. Rationalizing the denominator, limited to monomials and binomials of the second order, including such examples as:

$$\sqrt{\frac{x}{y}},\ \frac{\sqrt{3-x}}{\sqrt{2}},\ \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$$

Find the approximate value of $\dfrac{\sqrt{5}-1}{3-2\sqrt{5}}$ to three decimal places, *after* simplification.

6. Radical equations involving linear equations and quadratic equations having rational roots.

$$\sqrt{2x+3} = 2x-3$$
$$\sqrt{x-1} + \sqrt{x+4} = \sqrt{4x+5}$$
$$a = \sqrt{\frac{b}{x}}$$
$$\frac{5x-1}{\sqrt{5x}+1} = 1 + \frac{\sqrt{5x}-1}{1\frac{1}{3}}$$

Note.—All radical equations should be checked to insure that extraneous roots have not been introduced by squaring.

X. Imaginaries.

Note.—On entering upon the subject of imaginaries, it should be pointed out that negative numbers and fractions are not *real*

in the sense that numbers like 5 and $\frac{24}{6}$ are real, and hence in imaginaries the student is again extending his number system.

A. Definition equations.

1. $(\sqrt[n]{-a})^n = -a$

Corollaries:

a. $\sqrt{-a} = \sqrt{a}\sqrt{-1}$

b. $\sqrt{-a}\sqrt{-b} = -\sqrt{ab}$

2. $\sqrt{-1} = i$

1. Corollaries:

a. $i^{4q+1} = i$

b. $i^{4q+2} = -1$

c. $i^{4q+3} = -i$

d. $i^{4q+4} = +1$

Note.—Begin with $q = 0, 1, 2$, etc. The constant use of the symbol i is strongly recommended.

B. Complex numbers.

1. Definitions.

2. Theorems.

a. The sum and product of two conjugate complex numbers are both real.

b. If $X + iY = 0$, then $X = 0$ and $Y = 0$

Corollary:

If $X + iY = A + iB$, then $X = A$ and $Y = B$.

3. The fundamental operations.

a. Addition and subtraction.

b. Multiplication and division.

Rationalization of denominators.

Conjugates.

C. Graphs of complex numbers; their sums and differences.

XI. Quadratic Equations in One and Several Unknowns.

A. Equations in one variable.

1. Solution.

a. Incomplete.

(1) Pure, reducible to the form $x^2 = a^2$, $x = \pm a$; or $x^2 - a = 0$, $(x + \sqrt{a})(x - \sqrt{a}) = 0$, etc.

(2) Lacking the independent term, of form $ax^2 + bx = 0$. Solved by factoring.

b. Complete, reducible to the form $ax^2 + bx + c = 0$

Solved by 1. Factoring.

2. Completing the square.

3. Quadratic formula.

Note.—Fractional and radical equations which reduce to these type forms are included. Care should be taken that clearing of fractions or clearing of radicals does not introduce a false root.

c. Equations that are quadratic in a power of the variable or any function of it.

(1) In a power, as $ax^{2n} + bx^n + c = 0$

or $ax^{\frac{m}{n}} + bx^{\frac{m}{2n}} + c = 0$

(2) In any function, as

$$(x^2 + 2x)^2 - 2(x^2 + 2x) - 3 = 0,$$
$$2x^2 - 6x + \sqrt{x^2 - 3x + 6} - 9 = 0,$$
$$\frac{x^2}{x+1} + \frac{x+1}{x^2} = \frac{5}{2}.$$

Solved by factoring, or by substituting a new variable for the function involved.

d. Equations of higher degree that factor by known methods, especially by synthetic division.

2. Theory.

a. Proof of the quadratic formula.

b. Symmetric functions of the roots:

(1) Sum and product of the roots; application to testing results and to forming equations from given roots. Simple symmetric functions, as $\frac{1}{r} + \frac{1}{s}$, where r and s are the roots.

(2) Use of the product to show the roots are reciprocal, if the coefficient of the second degree term equals the independent term.

c. Discriminant test for the roots:

(1) Real or imaginary.

(2) Rational or irrational.

(3) Equal or unequal.

Note.—Applications such as the following are recommended: For what values of a will the roots of $2x^2 + (1 + a)x + 2 = 0$ be equal? Real?

d. Graphs. Type form, $ax^2 + bx + c$.

(1) Let $ax^2 + bx + c = y$.

(2) For approximation of the roots, change to form $x^2 + px + q$, let $y = x^2$ and $y + px + q = 0$. Plot the intersections of the curve and the straight line. The curve $y = x^2$ is convenient for all numerical quadratic equations.

e. Problems including numerical, geometrical, physical and commercial relations, provided their subject matter does not require extended explanation.

B. Equations in two variables that can be solved by quadratic methods.

1. Solution.

a. When one is of the first degree, the other of second degree or factorable by the simple equation so as to reduce to second degree.

Solved by substitution, as

$$ax + by = c$$
$$dx^2 + exy + fx = k, \text{ etc., etc.}$$

b. Homogeneous second degree equations.

(1) One entirely homogeneous, the other not, as

$$ax^2 + bxy + cy^2 = 0$$
$$lx^2 + y + my^2 = k$$

Solved by factoring, then substituting.

(2) Both homogeneous except for the independent term, as

$$ax^2 + bxy + cy^2 = d$$
$$ex^2 + fxy + gy^2 = k$$

Solved by eliminating the independent term, then treating like case (1).

c. Symmetric forms (or symmetric but for sign)

Solved by obtaining $x + y$ and $x - y$

(1) Of types $x+y$, $x-y$, xy, x^2+y^2,

$x^2 \pm xy + y^2$, $x^4 + x^2y^2 + y^4$, $\frac{x}{y} + \frac{y}{x}$

(2) Of type $(ax)^n + (by)^n = k$

$ax + by = 1$

Solved by elimination of the powers. Divide if possible.

d. Other methods.

(1) Divide one equation by the other.

(*a*) When it divides evenly, as

$$x^3 + y^3 = 28$$
$$x + y = 4$$

(*b*) When a like factor appears in a member of each equation, as

$$a(x^2 - y^2) = b$$
$$c(x - y) = d$$

(2) Add or subtract:

(*a*) When one unknown is thus eliminated, as

$$x^2 + y = 7$$
$$x^2 - x - y = 1$$

(*b*) When the result is a factorable form, as

$$x^2 + xy + y^2 + x + y = 10$$
$$xy + 2x + 2y = 8$$

(3) Factor when possible, as

$$x^2y^2 \pm 5xy + 4 = 0$$
$$x^2 + y^2 - xy = 13$$

e. Equations of the foregoing types which are in terms of functions of the variables, as

$$(x+y)^2 + x^2y^2 = 13$$
$$x + y - xy = 1$$

Solve for the functions (here, for $x+y$ and xy).

Such forms may be simplified by the substitution of a new variable.

2. Graphs.

Of any equation in two variables.

Approximation of results for cases that do not solve by quadratic methods. Simple cases only.

3. Problems.

Resulting in equations of the foregoing types.

C. Equations in three or more variables.
Simple types only.

XII. BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS.

A. Theory.
1. The derivation of the expansion of $(a+b)^n$ by simple inductive reasoning.
2. Discussion of the formula.
a. As a whole.
(1) Number of terms.
(2) Equality of coefficients of terms equally distant from the extremes.
b. The general term.
(1) Form of the coefficient in the fractional factored form.
(2) Sum of exponents in any term.

B. Applications.
1. To expansion of any binomial.
2. To finding any term, by formula or by analogy, as with the 3d term.
3. Such problems as, *a*. Find coefficient of x^{12} in $(x^2+2x)^{10}$
b. Find term independent of x in $\left(x^2-\frac{1}{x}\right)^{12}$
c. Find middle term of $\left(2-\frac{x}{3}\right)^8$
4. Powers of numbers, as $(1.1)^{12}=(1+.1)^{12}$ correct to 3 decimal places.

XIII. INEQUALITIES.

A. Recognition of the following principles:
1. Unequals combined with equals.
a. (1) Equals added to unequals give unequals in the same sense.
(2) Equals taken from unequals give unequals in the same sense.
(3) A term can be changed from one member of an inequality to the other if the sign of the term

be changed; application to annulment of like terms in both members.

(4) The signs of all terms of an inequality can be changed if the inequality sign is reversed.

b. (1) Positive equals multiplied into unequals give unequals in the same sense.

(2) Positive equals divided into unequals give unequals in the same sense.

c. (1) Unequals taken from equals give results unequal in the opposite sense.

(2) Positive unequals divided into positive equals give results unequal in the opposite sense.

2. Unequals combined with unequals.

a. Unequals added to unequals in the same sense give results unequal in that sense.

b. (1) Positive unequals multiplied by positive unequals in the same sense give results unequal in that sense.

(2) Positive unequals raised to a positive integral power give results unequal in the same sense.

(3) Positive unequals raised to a positive fractional power give results unequal in the same sense.

(4) Positive unequals raised to a negative power give results unequal in the opposite sense: reciprocals.

Note.—Unequals should not be taken from or divided by unequals, for the results can not in general be determined.

B. Solutions.

One unknown.

a. Of type $ax^2 + bx + c \gtreqless 0$

b. Of type $\frac{(x-a)(x-b)}{(x-c)(x-d)} \gtreqless 0$

Note.—Graphs can be used to show the values of x that make the function greater than, equal to, or less than, zero.

C. Use of the theorems, with the fact that a perfect square cannot be negative, to prove, (a and b being positive unequal numbers), $a^2 + b^2 > 2ab$, $a^3 + b^3 > a^2b + ab^2$, etc.

XIV. Ratio and Proportion.

A. Proofs of the following theorems:

1. *a.* The product of the means equals the product of the extremes.

   b. In a mean proportion, the mean equals the square root of the product of the extremes.

   c. Either mean equals the product of the extremes divided by the other mean, and either extreme equals the product of the means divided by the other extreme.

2. *a.* If the product of two quantities equals the product of two other quantities, either pair may be made the means, and the other pair the extremes, of a proportion.

   b. If four quantities are in proportion, any proportion of these quantities is true if the means of the original proportion are either both means or both extremes, in the new proportion.

3. *a.* If four quantities are in proportion, they are in proportion by composition.

   b. If four quantities are in proportion, they are in proportion by division.

   c. If four quantities are in proportion, they are in proportion by composition and division.

4. *a.* If four quantities are in proportion, equi-multiples (integral or fractional) of the antecedents are in proportion to equi-multiples of the consequents.

   b. If four quantities are in proportion, like powers of those quantities are in proportion.

5. If a series of ratios are equal, the ratio of the sum of their antecedents to the sum of their consequents equals any one of those ratios.

B. Applications.

1. Deduction of any proportion from a related proportion.
2. Finding one term if the others are known.
3. Solving equations (usually by composition and division).

XV. Progressions.

A. Arithmetic.

1. Deduction of formulas for

a. General or last term.
b. Sum of any number of terms.
c. Mean between two quantities.

2. Applications.
a. Given any three of the five numbers, first term, difference, number of terms, last term, sum, to find the other two.
b. Insertion of any number of means between two quantities.
c. Problems of other types that can be solved by use of the formulas.

B. Geometric.

1. Deduction of formulas for
a. General or last term.
b. (*a*) Sum of any number of terms.
(*b*) Limit approached by the sum of an infinite decreasing series.
c. Mean between two quantities.

2. Applications.
a. Given any three of the five numbers, first term, ratio, number of terms, last term, sum, to find the other two, unless the equations cannot be solved by known methods.
b. Insertion of any number of means between two quantities.
c. Evaluation of recurring decimals.
d. Problems of other types that can be solved by use of the formulas, some combining the two progressions, and some reviewing important types of quadratics.

General Notes.

1. All the topics in this syllabus may at first be studied in outline with simple examples and then reviewed in detail with harder examples.

2. There should be much practice in mental algebra. Students should learn to give the elementary, simple forms accurately and quickly.

3. Students should state principles in general language with-

out symbols and without writing; at other times the general principle should be given them and they should be required to translate it promptly into symbolic language.

4. The definition should be given at the moment that it is required. It should be more often evoked than imposed. The student may be allowed to make and to correct his own statement. A temporary definition may be allowed until such time as correction and enlargement are necessary.

5. Home examples should relate generally to what has already been studied in class. The student should be encouraged to select for himself examples belonging to any advance work he is preparing. Sometimes a special topic in which a student has made commendable progress may be assigned to that student for further work which he may explain to the class.

6. Clear, orderly expression, writing and arrangement should be exacted at all times. The student should be made to feel the disadvantage of disorderly arrangement in expression.

7. Guessing in answering should be discouraged. A student should only say what at least he believes to be correct, and he should not lay too much emphasis on answers merely as answers. He should distinguish between answers and roots in certain problems.

8. Great care should be taken that algebraic work does not become mechanical. It may be necessary to repeat many times the *meaning* of the various algebraic forms.

9. The equation is the chief concern of algebra. In performing the various operations on equations great care should be taken with signs, with interchange of members, with simplifications sometimes before and sometimes after clearing of fractions, etc. Care should be taken to distinguish between unknown and variable. In the simple graphs of equations with one unknown, and of simultaneous equations with two unknowns, very clear distinction of the lines representing the values of the unknown or unknowns should be made.

10. In literal equations the meanings of the several kinds of letters used should be emphasized and generously illustrated by numerical values.

11. Operations should be frequently checked, preferably by numerical substitution wherever possible.

ADVANCED.

LIST OF TOPICS.

I. LOGARITHMS.

A. THEORY.

1. The logarithm of 1 to any base is zero.

2. The logarithm of the base is 1.

3. The logarithm of the product of two numbers equals the sum of the logarithms of those numbers.

4. The logarithm of the quotient of two numbers equals the difference of the logarithms of those numbers.

5. (*a*) The logarithm of any power of a number equals the logarithm of the number multiplied by the exponent of the power. (*b*) The logarithm of any root of a number equals the logarithm of the number divided by the index of the root.

6. (*a*) Rule for finding the characteristic of any number using the base 10. (*b*) Use of tables; finding the logarithm of any number, and finding the number from the given logarithms, including interpolation.

B. APPLICATIONS.

1. Questions to be solved without tables; as, $\log_2 4 = ?$ $\log_8 \frac{1}{2} = ?$ If $\log_{10} 2 = .30103$, and $\log_{10} 3 = .47712$, find $\log_{10} 24$, $\log_{10} \frac{2}{3}$, $\log_{10} 32$, $\log_3 2$. What number, using the base 3, has a logarithm -3? If $\log_{10} 3 = .47712$, find how many places in 3^{200}. Write in form for logarithmic use $\log \dfrac{\sqrt{ac^2(b-c)}}{bc}$.

2. Use of logarithmic tables in performing arithmetic computations; as, 98.712^5, $\sqrt[7]{102.847}$, $\dfrac{\sqrt{432.9203}\times\sqrt{41.52}}{.024371}$, $\dfrac{2^{-\frac{1}{2}}\times(-7)}{300}$.

3. Solution of simple cases of exponential equations; as, $3^x = 7$, $3^{2x+3} = 2^{5x-7}$, $5^{x-y} = 3$ and $2^{3x+y} = 5$.

NOTE.—Management of a negative factor in logarithmic computation. Distinction between the logarithm of a negative number and a negative logarithm.

II. PERMUTATIONS AND COMBINATIONS.

A. THEORY.

1. (*a*) The number of ways in which two things can both be done is the product of the number of ways in which the first can be done by the number of ways in which the second can be done. (*b*) The number of ways in which one or the other of two things can be done is the sum of the number of ways in which each alone can be done.

2. ${}_nP_r = n(n-1)(n-2)\cdots(n-r+1)$, including ${}_nP_n = n!$.

3. $${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\ 2\ 3\cdots r} = \frac{n!}{r!\,(n-r)!} = {}_nC_{n-r}.$$

4. $${}_nP_n \text{ (where } k \text{ are of one kind and } l \text{ of another)} = \frac{n!}{k!\,l!}.$$

5. Simple combination formulas; as, ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$,

$${}_nC_1 + {}_nC_2 + {}_nC_3 + \text{etc.} \cdots + {}_nC_n = 2^n - 1.$$

B. APPLICATIONS; as,

Using the letters of the word "certain," how many arrangements of all the letters? of any five of the letters? of five of the letters of which two are vowels and the others consonants? of all the letters in a circle? of all the letters if *c* and *n* are the first and last? of all the letters if two vowels are first and last? of these letters and two other letters *a*? of all the letters if *r* and *t* can not be separated? of all the letters if *r* and *t* can not be together? How many sets of four letters can be chosen? of two vowels? of a vowel and three consonants? of five letters containing *a*? of five letters not containing *a*? If there are three ways to go from *A* to *B*, and two to go from *B* to *C*, how many to go from *A* to *C*? In how many ways can six people be seated in ten numbered seats? If ten teams in a league wish to play so that each team meets each other team twice, how many games are played? How many different throws can be made with two dice? etc.

NOTE 1.—Easy exercises in choice and chance (or probability) should be given.

NOTE 2.—Use the symbol ! or $\underline{|}$ for factorial in formulas.

III. COMPLEX NUMBERS.

NOTE.—This is a somewhat more extended treatment than that given in X of Elementary and Intermediate.

A. THEORY. (Using the general form $a + bi$.)

1. The sum or difference of complex numbers in the general form can be reduced to a complex number in the general form.

2. The product of complex numbers can be reduced to a complex number in the general form.

3. The quotient of two complex numbers can be reduced to a complex number in the general form.

4. The modulus; its definition and length; use in graphical representation (see X of Elementary).

B. APPLICATIONS.

To simplifying imaginary expressions, including such questions as, Extract the square root of $2 - 3i$; find and plot the three cube roots of 1; how long is the modulus of each? Simplify and represent graphically $\frac{3+2i}{5-3i}$ and of $\frac{i+7}{i-7}$, their sum and their difference, and test the results by adding and subtracting the complex numbers.

IV. DETERMINANTS.

A. THEORY.

1. If the rows of a determinant are changed to columns, and the columns to rows, the determinant is not changed in value.

2. (a) The interchange of two rows or of two columns changes the sign of the determinant. (b) If two rows or two columns are identical, the determinant equals zero. (c) If the substitution of one quantity for another in the constituents of a determinant (as, a for b) makes two rows or two columns identical, then the difference of those quantities (as, $a - b$) is a factor of the determinant.

3. (a) A factor of any row or of any column is a factor of the determinant. (b) If each constituent in a row or in a

column is the sum of n terms, the determinant can be expressed as the sum of n determinants. (c) Equimultiples of the constituents of any row (or of any column) can be added to, or taken from, the corresponding constituents of any other row (or column) without changing the value of the determinant.

4. (a) If the constituents of a row (or of a column) be multiplied, each by its co-factor, the sum of the products so obtained will equal the original determinant. (b) If the constituents of a row (or of a column) be multiplied, each by the co-factor of the corresponding constituent of a different row (or column), the sum of the products so obtained equals zero.

B. Applications.

1. Evaluation of a determinant: (a) of the third degree; (b) of any degree, by the use of minors; (c) of any degree, by adding or subtracting the constituents of one row (or column) from another so as to obtain zero constituents, and so to reduce the degree of the determinant.

2. The solution of simultaneous linear equations.

3. Finding the condition that n linear homogeneous equations in n variables have a common solution other than the zero solution.

4. Factoring determinants by the method $A1(c)$.

Note.—Introduce the subject by the solution of a set of linear equations in two variables. Numerical examples.

V. THEORY OF EQUATIONS.

Note.—Review factor theorem, remainder theorem, synthetic division. See II, C, notes 1 and 2.

A. General Theory.

1. An equation of the nth degree has n roots. (Assume that any equation has at least one root.)

2. (a) Relation between coefficients and roots; simple cases of symmetric functions of the roots; as, find the sum of the reciprocals of the roots of a given equation. (b) Application to (1) depression of an equation; as, if one root of a given equation of third degree is r, write the quadratic equation having its other two roots; (2) formation of an equation having given roots; as, $a \pm bi$, c; 2, 3, -4, 5.

3. (*a*) If the coefficient of the highest power of the variable in an equation is one, and the other coefficients are integral, the equation can not have a rational irreducible fraction as a root. (*b*) If an equation has rational coefficients, quadratic surd roots occur in conjugate pairs: if an equation has real coefficients, imaginary or complex roots occur in conjugate pairs.

4. Transformation of an equation by: (*a*) Multiplying the roots by a constant; including changing the signs of the roots by using -1; as, multiply the roots of $2x^3+3x^2-4x+7=0$ by 3; change the signs of its roots; change it into an equation having the coefficient of x^3, 1, and the other coefficients integral; divide each root by 3. (*b*) Increasing or diminishing each root by a given constant; also, by means of this, making the coefficient of the second highest power of the variable equal to zero; as, in the equation in (*a*), add 2 to each root; take three from each root; make the coefficient of x^2 zero.

B. Graphical Representation of Functions.

1. Graphs of functions of one variable.

2. Interpretation of the graphs, including: (*a*) finding integral roots of the function, (*b*) locating and approximating rational fractional and surd roots, (*c*) finding those values of the variable that give positive or negative values of the function, (*d*) approximating those values of the variable that give maximum or minimum values of the function.

C. Character and Location of the Roots.

1. Descartes' rule of signs (without proof). Its use in finding the possible number of positive and negative roots, as in $x^3+x^2-3x+5=0$, and in finding the exact number of positive, negative and imaginary roots; as, find the character of the roots of $x^3+x-3=0$.

2. Location of the roots by the signs of the remainders obtained by substitution or division. Special importance should be given to the substitution of $-\infty$, 0, and $+\infty$.

3. By graps; see B, 2 (*b*).

D. Solution of Equations.

1. Commensurable roots, by substitution or division.

2. Incommensurable roots, by Horner's method of approximation.

NOTE.—The finding of incommensurable roots should not be made unnecessarily burdensome by carrying the work to a useless number of decimal places. Two decimal places is probably enough for most examples of this kind.

NOTE.—While the topic of mathematical induction is not included in this syllabus, it may be desirable to give some idea of that method of reasoning and to apply it to the binomial theorem and perhaps to other simple investigations.

The committee is fully persuaded that no other topics of algebra should be required of secondary school pupils at this time, and it wishes to call the attention of both colleges and preparatory schools to the unfortunate and harmful lack of reasonable uniformity in the requirements of the colleges in advanced algebra. If the committee is justified in the belief that the topics included in this syllabus provide a sufficient preparation for the needs of college work, then the colleges should adjust their entrance requirements in algebra as near as may be to this syllabus.

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DEVELOPMENT OF THE FUNDAMENTAL IDEAS OF THE DIFFERENTIAL CALCULUS.

By G. H. Graves.

The purpose of this paper is to present the characteristic features of the different methods, by which the differential calculus has been introduced to students beginning the study of the subject.

The *first* traces of the science are to be found among the Greeks. In passing from commensurable to incommensurable figures, from rectilinear to curvilinear, the Greek mathematicians were obliged to consider a process repeated indefinitely which gave rise to quantities indefinitely small. The ancient geometers surmounted this difficulty by introducing the "Method of Exhaustions," which may be illustrated by the following theorem from Euclid:

"Circles are to one another as the squares on their diameters."*

Let C and C' be two circles and D and D', their diameters. Also let L and G be any two areas such that $L < C'$ and $G > C'$. Then

$$C : L \neq D^2 : D'^2,$$

because we may inscribe in C' a regular polygon $P' > L$, and if P is the similar polygon inscribed in C,

$$P : P' = D^2 : D'^2,$$

where $\quad P < C$ and $P' > L$.

Also

$$C : G \neq D^2 : D'^2,$$

because we may circumscribe about C' a regular polygon $R' < G$, and if R is the similar polygon circumscribed about C,

$$R : R' = D^2 : D'^2,$$

where $\quad R > C$ and $R' < G$.

* Heath's translation (Cambridge, 1908) Book xii., Prop. ii. The proof above is merely a summary of Euclid's.

Hence

$$C : C' = D^2 : D'^2.$$

This is essentially what we still use in elementary geometry and is mathematically rigorous.

The *second* method of dealing with such problems is known in its present form as the "Method of Infinitesimals."* The beginnings of this method are seen in the work of Kepler† and Cavalieri.‡ These men considered a curve as a polygon with an infinite number of sides and based their deductions on this paradoxical substitute for the precise idea that a curve is the limit of the inscribed polygons as the number of their sides is increased. The culmination of this movement was seen in the work of Leibnitz.

He is chiefly interested in a "differential," dx, which he obtains on the metaphysical assumption that successive division results at last in something which can be no further subdivided. His dx is one of these indivisible parts of the x-axis, not zero, yet so small that two finite quantities differing by dx are to be regarded as interchangeable. In his letter to Wallis, March 30, 1699, he says:§

"It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero, but which are rejected as often as they occur with quantities incomparably greater. Thus if we have $x + dx$, dx is rejected. But it is different if we seek the difference between $x + dx$ and x, for then the finite quantities disappear. Similarly we cannot have xdx and $dxdx$ standing together. Hence if we are to differentiate xy we write:

$$(x + dx)(y + dy) - xy = xdy + ydx + dxdy.$$

But here $dxdy$ is to be rejected as incomparably less than $xdy + ydx$. Thus in any particular case the error is less than any finite quantity."

* Examples of text-books using this method: Lodge, McMahon and Snyder.

† "Rechnung der Cörperlichen Figuren" (1616).

‡ "Geometria Indivisibilibus Continuorum" (1635).

§ "Leibnitzens Mathematische Schriften," Gerhardt ed., Vol. 4, p. 63 (Series III. in Leibnitzens Gesammelte Werke, Pertz ed., Halle, 1859).

This neglecting of quantities which, however small they be, are expressly declared not to be zero, is of course inadmissible in mathematics except as a method leading to results confessedly approximate. The philosopher Berkeley, while he did not challenge the accuracy of the results obtained by this method, called its loose reasoning severely to account.* He showed in a few cases that the correctness of results was due to compensating errors, and suggested that this is always the case. Others have carried out this suggestion in greater detail.†

The *third* method, that of "Fluxions," was discovered by Newton at about the same time as the one just mentioned. With a little explanation as to what is meant by velocity at a point, it has become our "Method of Rates."‡ Newton considered a quantity in a state of change, *e. g.*, a curve, as it is being described by a moving point. He called the infinitesimal path traced in the infinitesimal time the "moment" of the "flowing quantity" (*i. e.*, of the variable), and the ratio of the moment to its corresponding time, the "fluxion" of the variable (*i. e.*, the velocity of the motion). He denoted the fluxion of x by $\dot{x}$. Hence he can state:§ "The moments of flowing quantities are as the velocities of their flowing or increasing," which we should express:

$$\frac{dy}{dx} = \frac{dy}{dt} : \frac{dx}{dt}.$$

His use of moments and fluxions is seen in the following explanation from the Method of Fluxions (pp. 24, 25):

"If the moment of x be represented by the product of its celerity $\dot{x}$ into an indefinitely small quantity o (that is $\dot{x}o$), the moment of y will be $\dot{y}o$, since $\dot{x}o$ and $\dot{y}o$ are to each other as $\dot{x}$ and $\dot{y}$. Now since the moments as $\dot{x}o$ and $\dot{y}o$ are the indefinitely little accessions of the flowing quantities, x and y, by which these quantities are increased through the several indefinitely little intervals of time, it follows that those quantities, x and y, after any indefinitely small interval of time, become

* *The Analyst*, London, 1734.

† *E. g.*, Carnot, "Reflexions sur la metaphysique du Calcul infinitesimal" (Paris, 1801).

‡ Text-books: Hayes (1704), Rice and Johnson, Hardy.

§ "Method of Fluxims," translated by Colson (London, 1736), p. 24.

$x + \dot{x}o$ and $y + \dot{y}o$. And therefore the equation which at all times indifferently expresses the relation of the flowing quantities will as well express the relation between $x + \dot{x}o$ and $y + \dot{y}o$ as between x and y; so that $x + \dot{x}o$ and $y + \dot{y}o$ may be substituted in the same equation for those quantities instead of x and y.

"Therefore let any equation

$$x^3 - ax^2 + axy - y^3 = 0$$

be given, and substitute $x + \dot{x}o$ for x and $y + \dot{y}o$ for y, and there will arise

$$\begin{aligned} &x^3 + 3x^2\dot{x}o + 3x\dot{x}\dot{x}oo + \dot{x}^3o^3 \\ &- ax^2 - 2ax\dot{x}o - a\dot{x}^2oo \\ &+ axy + ax\dot{y}o + a\dot{x}oy + a\dot{x}o\dot{y}o \\ &- y^3 - 3\dot{y}oy^2 - 3\dot{y}^2ooy - \dot{y}^3o^3 = 0. \end{aligned}$$

Now by supposition,

$$x^3 - ax^2 + axy - y^3 = 0,$$

which therefore being expunged and the remaining terms being divided by o, there will remain

$$3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3oo - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x \\ + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3oo = 0.$$

But whereas o is supposed to be infinitely little that it may represent the moments of quantities, the terms which are multiplied by it will be nothing in respect to the rest. Therefore I reject them and there remains:

$$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2."$$

This is open to the same criticism as Leibnitz's work. Newton himself however improved upon it in his method of "first and last ratios."

This *fourth* method is now known as the "Method of Limits." Newton's use of it will appear in the following example from the introduction to his "Quadrature of Curves."*

"Let a quantity x flow uniformly and let it be required to find the fluxion of x^n. In the time in which x by flowing becomes

* Tractatus de Quadratura Curvarum (London, 1704), Introduction.

$x+o$, the quantity x^n becomes $\overline{x+o}|^n$; *i. e.*, by the method of infinite series,

$$x^n + nox^{n-1} + \frac{nn-n}{2}oox^{n-2} +, \text{etc.},$$

and the increment o and

$$nox^{n-1} + \frac{nn-n}{2}oox^{n-2} +, \text{etc.},$$

are to each other as 1 and

$$nx^{n-1} + \frac{nn-n}{2}ox^{n-2} +, \text{etc.}$$

Now let the increment vanish and their last ratio will be 1 to nx^{n-1}."

In the same introduction he gives the familiar geometric interpretation of these ratios as the slopes of a secant through two points of a curve and of the tangent which it approaches as the two points approach coincidence. He remarks:

"If the points are distant from each other by an interval however small, the secant will be distant from the tangent by a small interval. That it may coincide with the tangent and the last ratio be found, the two points must unite and coincide altogether. In mathematics errors, however small, must not be neglected."

Newton was criticized* because he here keeps expressions got by contradicting the assumption on which the expressions were first obtained. Also as Lagrange observes:† "This method has the great inconvenience of considering quantities in the state where they cease, so to say, to be quantities." And in general the employment of vanishing quantities in this manner confuses the limit of a variable ratio in which both terms approach zero simultaneously, with the ratio of two zeros. It is clear that Newton himself understood the difference, however, for he says, in his "Principia":‡ "Ultimate ratios in which quantities vanish, are not, strictly speaking, ratios of ultimate quantities, but limits to which the ratio of these quantities decreasing without limit, approach, and which, though they can come nearer than any given difference whatever, they can neither pass over nor attain before the quantities have diminished indefinitely."

* *E. g.*, by Berkeley in "The Analyst."

† "Théorie des Fonctions" (Paris, 1813), Introduction.

‡ "Philosophiæ Naturalis Principia Mathematica," Section I.

With this idea of limits, the differential calculus finds a starting point from which it may be rigorously developed, and this is the method generally followed in modern text-books.*

Lagrange objected to this method on the ground that it requires limits to be regarded "in a peculiar manner, if not contrary, at least foreign to the spirit of modern analysis."† He proposed a *fifth* plan of introducing the calculus by the purely algebraic means of "derived functions."

In referring to the work of his predecessors he says:‡ "When we consider these different methods, we find that their only aim is to give the means of obtaining separately, the first terms of the development of a function, and in detaching and isolating these, so to say, from the rest of the series." His plan may be briefly set forth in his own words:§

"When to the variable of a function we give any increment, we may, by the ordinary rules of algebra expand the function, if it is algebraic, according to the powers of the increment. The first term of the expansion will be the given or 'primitive' function; the following terms will be formed of different functions of the same variable multiplied by successive powers of the increment. These new functions depend only on the primitive function from which they are derived and may be called 'derived functions.' In general whatever be the primitive function, algebraic or not, it may always be expanded or considered to be expanded in the same manner and thus give rise to derived functions."

In many ways this is a simplification, but to be rigorous as a general method it requires the consideration of the problem whether all functions are developable into power series, and whether the series obtained are convergent. This requires limits again "regarded in a peculiar manner." But in those functions which expand into a finite series or an infinite series which is easily tested, this method is useful and is often used in text-books on algebra.‖

Finally we must mention Cauchy who has placed the Method

* *E. g.*, Todhunter (1864), Byerly, Osgood.

† Introduction to the Leçons sur le Calcul des Fonctions (1808).

‡ Ibidem.

§ Théorie des Fonctions, Introduction. (Paris, 1813.)

‖ *E. g.*, Fine, Hall and Knight.

of Infinitesimals on a sound basis and relieved modern analysis of the necessity of being cumbered with $\frac{dy}{dx}$ as an inseparable ratio.

He states in the preface to his "Leçons sur le Calcul Infinitesimal": "My principal aim has been to reconcile rigor with the simplicity that results from the direct consideration of infinitely small quantities." This he does by defining an infinitesimal as a variable which has zero for its limit. He makes the transition from

$$\frac{dy}{dx}=\frac{f(x+i)-f(x)}{i}=f'(x)$$

to

$$dy=f'(x)dx$$

as follows:*

"Let $y=f(x)$ be a function of the independent variable x; i, an infinitesimal, and h a finite quantity. If we put $i=\alpha h$, α will be an infinitesimal and we shall have the identity

$$\frac{f(x+i)-f(x)}{i}=\frac{f(x+\alpha h)-f(x)}{\alpha h},$$

whence we derive

$$(1)\qquad \frac{f(x+\alpha h)-f(x)}{\alpha}=\frac{f(x+i)-f(x)}{i}h.$$

The limit toward which the first member of this equation converges when the variable α approaches zero, h remaining constant, is what we call the "differential" of the function $y=f(x)$. We indicate this differential by the characteristic, d, as follows:

$$dy \text{ or } df(x).$$

It is easy to obtain its value when we know that of the derived function, y' or $f'(x)$. In fact taking the limits of both members of equation (1), we have in general :

$$(2)\qquad df(x)=hf'(x).$$

In the particular case where $f(x)=x$, equation (2) reduces to

$$dx=h.$$

* "Résumé des Leçons sur le Calcul Infinitesimal," Quatrième Leçon (Paris, 1823). [Œuvres Complètes, Ser. II., Tome IV., Paris, 1899.]

Thus the differential of the independent variable, x, is simply the finite constant, h. Substituting, equation (2) will become

$$df(x) = f'(x)dx,$$

or what amounts to the same thing

$$dy = y'dx."$$

The plan generally followed in text-books to-day is not to introduce infinitesimals until a clear concept of the derivative has been obtained from the Method of Limits. Owing to the greater ease of operating with infinitesimals, and their use in higher analysis, the beginner needs to gain familiarity with them, but at the outset even simplicity of operations must give place to clearness of concept and here the Method of Limits seems to offer the least difficulties. It will be observed that all the methods we have mentioned depend ultimately on limits. When the student has learned to associate with the derivative, $f'(x)$ or $D_x u$, the concrete idea of the slope of a tangent to a curve it is not hard to conceive of dx as a finite increment to the abscissa of a point, and dy as the segment cut off by the tangent on the increment to the ordinate. "The principal aim," says Professor Klein,* "is to convince the student that there is here nothing mysterious, but only simple things which every one can understand."

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* "Elementarmathematik vom Höheren Standpunkt aus" (Leipzig, 1908), p. 488.

NEW BOOKS.

Elementary Geometry, Practical and Theoretical. By C. GODFREY and A. W. SIDDONS. Cambridge: The University Press. Pp. 311. G. P. Putnam's Sons, New York Agents.

The authors of this book were among the leaders of the reform for the improvement of geometrical teaching in England, and the work has a clearness and vigor that will appeal to teachers. Part I. is devoted to experimental geometry and the many concrete examples will show the wide applications of the subject. This new edition contains an appendix on the pentagon group of constructions.

Modern Geometry. By C. GODFREY and A. W. SIDDONS. Cambridge: The University Press. Pp. 162. G. P. Putnam's Sons, New York Agents.

This book is a sequel to the "Elementary Geometry," by the same authors. The scope of the book seems to be limited to suit the needs of special classes of students in England. It will, however, be found a good introduction to more advanced and comprehensive works on the subject. It contains a large number of well-chosen exercises.

Mechanics. By JOHN COX. Cambridge: The University Press. Pp. 345. G. P. Putnam's Sons, New York Agents.

The author follows the historical development of the subject and the influence of Mack, to whom the book is dedicated, is seen throughout.

The work is divided into four books as follows: Book I., The Winning of the Principles; Book II., Mathematical Statement of the Principles; Book III., Applications to Various Problems; Book IV., Elements of Rigid Dynamics. It seems especially well suited to students of elementary physics and would be an excellent introduction to more advanced treatments as well as very suitable for those who do not take more advanced work.

The Elements of the Theory of Algebraic Numbers. By L. W. REID. New York: The Macmillan Company. Pp. 461. $3.50 net.

The author of this book has given a rather clear introduction to the general theory of algebraic numbers. The rational realm and the general quadratic realm have been treated from the point of view of the general realm of the *n*th degree so that the proofs given are readily extendable and the student will find it a valuable introduction and preparation for the more advanced works if he wishes to go on with the subject. The need of such a work has been long felt in this country and it is to be hoped it will find its way to many hands and receive the attention it deserves.

Elementary Analysis. By PERCEY F. SMITH and WILLIAM A. GRANVILLE. Boston: Ginn and Company. Pp. 223. $1.50.

This volume presents a course of seventy lessons in mathematics beyond trigonometry. In constructing it the authors have had in mind the needs of those students in natural and experimental science for whom, at the present day, a thorough understanding of coördinates, function, graph, rate, and summation is indispensable, but to whom the portions of mathematics which are formally difficult are less important.

The opening chapters deal with topics from analytic geometry, with numerous and varied applications, sufficient to give the student a grasp of the essentials of the subject.

A chapter on Functions gives the student practice in setting up functions, and teaches what may be learned of their nature by construction of their graphs only. The problems of this chapter cover a wide range and have been selected primarily with the idea of interesting the student.

A chapter on Curve Plotting is devoted to exponential, logarithmic, and trigonometric curves, with careful explanations of the properties of the corresponding functions and with emphasis on the principle of the *addition of ordinates*. Simple discontinuities also are considered in this chapter.

Principles of Secondary Education. Volume III. Ethical Training. By CHARLES DE GARMO. New York: The Macmillan Company. Pp. 223. $1.00 net.

The object of this book as stated by the author is to bring into clearer light the moral functions of knowledge and to show how over the bridge of ethical interest youth may be led from understanding to wisdom, and from wisdom to its correlative goodness and volitional efficiency. Another purpose is to make clear the great existing differences in ideals and conditions between the ancient static and socialistic organizations under an economy of deficit and pain, with their resultants of struggle and sacrifice, and the modern dynamic and democratic order under an economy of surplus and satisfaction with their resultants of personal independence and coöperative well being. One idea animates the whole; namely, that there may be realized the completest possible utilization of the agencies for ethical training now available to the American high school.

Analytic Geometry. By N. C. RIGGS. New York: The Macmillan Company. Pp. 300. $1.00 net.

This book differs somewhat from the usual text-books on the subject in that it devotes relatively less space to conics and more to such topics as trigonometric and exponential functions, parametric equations, maxima and minima, and graphic solution of equations. It aims to furnish a natural introduction to calculus as well as to bring out the fundamental principles and methods of the subject.

Elements of Plane and Spherical Trigonometry (With Tables). By DAVID A. ROTHROCK. New York: The Macmillan Company. Pp. 158. $1.40 net.

Practical Business Hints. By MARY H. WOODBURY. Salem: Newcomb and Gauss. Pp. 116.

The author aims in this book to teach by a series of lessons that practical business knowledge that every one should know. It contains a review of the fundamental operations with integers and fractions and treats of percentage, interest, discount, commission, taxes, accounts, banks and banking, business letters, notes, etc.

Applications of the Calculus to Mechanics. By E. R. HEDRICK and O. D. KELLOGG. Boston: Ginn and Company. Pp. 116.

This book is an endeavor to fill a need felt at some institutions of getting students better acquainted with the methods of applying the calculus to problems in mechanics. It aims at teaching them how to give the problem an analytic formulation and how to interpret the analytic results. It seems very well suited to these purposes and the student that uses it will know more calculus as well as more mechanics.

An Elementary Course in Graphic Mathematics. By MATILDA AUERBACH. Boston: Ally and Bacon. Pp. 54.

First Course in Algebra. By HERBERT E. HAWKES. Boston: Ginn and Company. Pp. vii + 334. $1.00.

The "First Course in Algebra" is designed for the first year's work. The topics considered have been strictly limited to those which belong primarily to study in the first year. Many helpful suggestions, the fruit of the widespread discussion of mathematical teaching which has marked the progress of the past ten years, are embodied in the book.

Difficult exercises have been avoided. Those given are new, varied, graded with extreme care, and amply sufficient to develop the essentials of elementary technic. The principles, so far as possible, are developed from the student's knowledge of arithmetic. An abundance of typical solutions are given, and rules based on them have been carefully stated in full. Wherever practicable, suitable methods of checking are illustrated.

Unusual emphasis is placed on problem work, especially in developing the student's ability to express a problem in terms of algebraic symbols. Variety is secured and interest maintained by frequent changes from technical work to problem work.

Close and consistent correlation with geometry is secured in two ways: through the problems based on elementary geometrical theorems, and by treating fully those radical forms which arise in geometry.

Graphs are used freely and are always incorporated in the work of the topic they are intended to illustrate.

Historical and biographical notes add a touch of human interest to the subject, and several portraits of famous mathematicians are included at points in the text where their work has contributed to the development of the subject.

Plane Geometry with Problems and Applications. By H. E. SLAUGHT, Associate Professor of Mathematics in the University of Chicago, and N. J. LENNES, Instructor in Mathematics in Columbia University. Published by Allyn and Bacon.

This book is sufficiently different from the numerous editions of "geometry" to deserve special attention. To quote from the preface: "The subject has been enriched by including many applications of special interest to the pupils. . . ." Free use is made of certain sources of problems which may be easily comprehend without extended explanations—such problems pertain to decoration, ornamental designs and architectural forms." On closer investigation it appears that these applications are intended merely to furnish a *concrete setting* for the usually too abstract geometrical theorems. There is no claim made that they are practical in the sense that the study of bookkeeping is practical for one who intends to keep books. The aim is to present the subject "so that the pupils may learn to know the essential facts of elementary geometry as properties of the space in which they live and not merely as statements in a book."

The large number of finely executed cuts give the book an unusual and particularly attractive appearance.

Throughout the book there is a large number of simple exercises, many of which can be answered at sight. Algebraic solutions are used freely, and manipulation of algebraic expressions such as radicals is required in an unusually large and varied number of problems. Many are exceedingly simple, such as; Ex. 30, p. 150, while others are quite difficult such as Ex. 18, p. 266.

While the logical rigor of the older texts has been fully maintained and in some cases improved (see the treatment of incommensurables), the whole subject has been recast with the avowed purpose of adapting to the needs and powers of the pupil. It is interesting to note that while this book and the new "Syllabus in Geometry for the State of New York" were unquestionably prepared entirely independently of one another, the book carries out the letter and spirit of that syllabus as fully as if it were made with special reference to it. There seem therefore to be some reasons for believing that recent discussions on the pedagogy and subject matters of geometry have brought into clearer light certain fundamental principles which are being generally accepted.

This book is bound to exert a wide influence on the teaching of geometry and to other texts which will appear in the future.

The "Solid Geometry" by the same authors, which is expected soon, will receive a cordial welcome from teachers who have already looked over the "Plane Geometry."

NOTES AND NEWS.

THE autumn meeting of the Philadelphia Section of the Association of Teachers of Mathematics in the Middle States and Maryland was held on Thursday afternoon, October 27, 1910, in the Auditorium of Houston Hall, University of Pennsylvania, the chairman, Professor George H. Hallett, presiding.

The announcement that Dr. Brumbaugh, Superintendent of the Public Schools of Philadelphia, would talk on "The Course of Study in Arithmetic in the Public Schools of Philadelphia," brought to the meeting, besides the members, several hundred of the teachers of the elementary schools of Philadelphia, who were anxious to hear something of the new course in arithmetic just being put into operation this fall. Dr. Jacobs, associate superintendent, elaborated somewhat on Dr. Brumbaugh's remarks.

The discussion was opened by Professor I. J. Schwatt, of the University of Pennsylvania. Mr. J. Eugene Baker, principal of the Philadelphia High School for Girls, also entered into the discussion.

Resolutions upon the death of one of the members, Dr. George B. McClellan Zerr, were adopted. Dr. Zerr was a member of the faculty of the Central Manual Training School.

The officers of the Philadelphia Section for the academic year 1910–11 are: *Chairman*—Professor George H. Hallett, University of Pennsylvania; *Vice-chairman*—Dr. Edward D. Fitch, DeLancey School; *Secretary*—Miss Elizabeth B. Albrecht, Philadelphia High School for Girls; *Members of the Executive Committee*—Miss Agnes Long, William Penn High School for Girls; Mr. Harry Rothermel, High School for West Philadelphia Boys.

The mid-winter meeting of the Philadelphia Section will take the form of a dinner to be held on Saturday evening, February 18, 1911.

Resolution on the Death of Doctor Zerr, Adopted by the Philadelphia Section, October 27, 1910.

Whereas, The Philadelphia Section of the Association of Teachers of Mathematics in the Middle States and Maryland has learned with profound regret of the death of George B. McClellan Zerr,

Therefore be it Resolved: That in the death of Doctor Zerr, the Philadelphia Section has lost a member highly esteemed for his depth of scholarship, for his enthusiasm and skill as a teacher, and for the many friendly qualities that won the respect of all with whom he was associated.

Resolved: That these resolutions be spread upon the minutes of the Section; that a copy be forwarded by the Secretary to the bereaved family as an expression of our sympathy; and that they be published in The Mathematics Teacher, the organ of the Association.

The fifteenth meeting of the association was held in the lecture room of the chemistry building of the University of Pennsylvania on Saturday, November 26. Dr. Edgar F. Smith, provost-elect of the university, made the address of welcome, which was replied to by the president, Dr. Metzler.

The proposed amendment was given its final reading and was unanimously passed.

Amendment to Article 7 of the Constitution: "The annual dues shall be fifty cents for subscribers to The Mathematics Teacher, and sixty cents to non-subscribers. All dues are payable at the time of the annual meeting."

The council passed a motion recommending that the association withdraw from membership in the American Federation of Teachers of the Mathematical and the Natural Sciences, on account of its heavy cost in proportion to our dues, and the doubtful benefit being derived from it. The association referred the matter to the council for final action, giving them full power to withdraw if it seemed best to do so after further consideration.

The following officers were elected for the year:

President—William H. Metzler, Syracuse University.

Vice-President—Philip R. Dean, Curtis High School, Staten Island, N. Y.

Secretary—Howard F. Hart, Montclair High School, Montclair, N. J.

Treasurer—Mrs. Clara H. Morris, High School for Girls, Philadelphia, Pa.

Members of the Council—Eugene Randolph Smith, Polytechnic Prepartory School, Brooklyn; Clifford B. Upton, Teachers College, New York; Fletcher Durrell, The Lawrenceville School, Lawrenceville, N. J.

Associate Editor of The Mathematics Teacher.—G. Alvin Snook, Central High School, Philadelphia, Pa.

The following reports of finances were given, and after auditing, were approved.

Report of Treasurer, November 26, 1910.

Receipts.

Balance December 13, 1909	$254.29
Dues December, 1909–November 23, 1910	462.85
Total receipts	$717.14

Disbursements.

Mathematics Teacher		$225.00
Philadelphia Section		36.75
Syracuse Section		25.00
New York Section		25.00
Rochester Section		3.00
Syllabus committee		18.85
General association and	$67.85	
Secretary's Office	40.22	108.07
Treasurer's Office		48.80
Total disbursements		$490.47

Total receipts	$717.14
Total disbursements	490.47
Balance on hand	$226.67

Approved, D. D. Feldman,
Arthur Sullivan Gale,
Auditing Committee.

Report of The Mathematics Teacher.

Receipts.

Balance, November, 1909	$ 44.93
From treasurer	225.00
Advertising; subscriptions	365.01
	$634.94

Disbursements.

Printing	$543.23
Postage, other expenses	44.15
Advertising	25.00
Balance, November, 1910	22.53
	$634.94

It was decided that the next meeing should be held at New York, the choice of the institution being left to the president and secretary. The next annual meeting is to be held in conjunction with The Association of Colleges and Preparatory Schools of the Middle States and Maryland.

The report of the committee on continuation schools, Mr. Breckenridge, chairman, was accepted as a report of progress, and the committee was continued.

The report of the algebra syllabus committee was discussed and was adopted as amended; it will be found in full in THE MATHEMATICS TEACHER. The committee was continued with a vote of thanks and instructions to work for the adoption of the report.

The rest of the meeting was given to the reading and discussion of the papers. The morning session was taken up in discussing "Is the Average Secondary School Pupil Able to Acquire a Thorough Knowledge of All the Mathematics Ordinarily Given in these Schools?"

The paper was read by Isaac J. Schwatt, and was discussed by Rev. James J. Dean, Edward D. Fitch, E. B. Ziegler and by numerous others. The afternoon paper, on "Training for Efficiency in Elementary Mathematics," was read by Ernest H. Koch.

A very pleasant feature of the meeting was the luncheon to which the University of Pennsylvania invited all those present at the meeting. The luncheon was served during the noon intermission in the auditorium of Houston Hall, which was very attractively decorated.

NEW MEMBERS.

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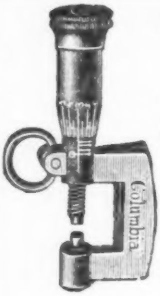

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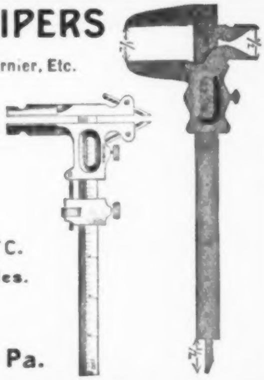

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